

NONNEGATIVE HERMITIAN HOLOMORPHIC VECTOR BUNDLES AND CHERN NUMBERS

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Dedicated to Professor Boju Jiang on the occasion of his 80th birthday

ABSTRACT. We show in this article that if a holomorphic vector bundle over a compact complex manifold has a nonnegative curvature Hermitian metric in the sense of Bott and Chern, which always exists on globally generated holomorphic vector bundles, then some special linear combinations of Chern forms with respect to the Chern connection are nonnegative. This particularly implies that all the Chern numbers of such a holomorphic vector bundle are nonnegative and can be bounded below and above respectively by two special Chern numbers in an explicit manner. As applications, we obtain a family of new results on compact complex manifolds whose holomorphic tangent or cotangent bundles are globally generated, some of which improve several classical results.

1. INTRODUCTION

The concept of positivity/nonnegativity has played a central role in complex differential geometry and algebraic geometry. Bott and Chern ([BC65]) introduced a notion of nonnegativity on Hermitian holomorphic vector bundles over compact complex manifolds and applied it to initiate the study of high-dimensional value distribution theory. As showed in [BC65], the existence of such a metric implies that the Chern forms are all nonnegative and on globally generated holomorphic vector bundles there always exist admit such metrics. Later, globally generated holomorphic vector bundles were investigated in detail by Matsushima and Stoll in [MS73] and they, among other things, related the non-vanishing of the Chern classes and Chern numbers to the transcendence degree of meromorphic functions on the base manifolds.

Soon after the appearance of [BC65], the notions of ampleness and Griffiths-positivity were introduced respectively by Hartshorne ([Ha66]) and Griffiths ([Gr69]) and it turns out that Bott-Chern nonnegativity implies Griffiths-nonnegativity (cf. Example 3.4) while Griffiths positivity implies ampleness (Note that ampleness was called cohomological positivity in [Gr69]). Griffiths raised in [Gr69] the question of characterizing the polynomials in the Chern classes/forms which are positive as cohomology classes/differential forms for Griffiths-positive or ample vector bundles. On the class level this was answered completely by Fulton and Lazarsfeld ([FL83]), extending an earlier result of Bloch and Gieseker ([BG71]). Indeed Fulton and Lazarsfeld showed that the set of such polynomials in the Chern classes for ample vector bundles is exactly the cone generated by Schur polynomials of the Chern classes.

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These inequalities of Fulton-Lazarsfeld type were showed by Demailly, Peternell and Schneider ([DPS94]) to remain true for numerically effective (“nef” for short) vector bundles over compact Kähler manifolds.

On the other hand, on the form level Griffiths’ question is still largely unknown. Griffiths himself showed that the second Chern form is positive on a Griffiths-positive holomorphic vector bundle. Although in [Gr69, p. 249] the proof is only for rank 2 vector bundles over surfaces, it can be easily adapted to the general case. Also Note that its proof is purely algebraic, which seems to be difficult to be generalized to higher dimensions. Recently Guler ([Gu12]) showed that the dual Segre forms, which are formal inverse of the total Chern forms, are positive on Griffiths positive vector bundles using some geometric arguments.

The main purposes of this article are two-folded. Our *first main purpose* is to show that, on a Bott-Chern nonnegative Hermitian holomorphic vector bundles, the cone generated by the Schur polynomials in the Chern forms are nonnegative (Theorem 2.3), thus providing some positive evidence to the aforementioned Griffiths’ question on the form level. Our proof of Theorem 2.3 is built on a relationship established in [FL83] that the cone generated by the Schur polynomials is contained in a cone defined by Griffiths in [Gr69]. When taking some special Schur polynomials in Theorem 2.3, we shall see that the self-products of Chern forms and thus all the Chern numbers of Bott-Chern nonnegative Hermitian holomorphic vector bundles can be bounded below and above by two of them respectively in an explicit manner (Theorem 2.4). As we shall see later in Example 3.1 that globally generated holomorphic vector bundles admit Bott-Chern nonnegative Hermitian metrics, this implies that Theorem 2.4 imposes strong constraints on the Chern numbers of such vector bundles. So our *second main purpose* is to apply Theorem 2.4 to such holomorphic vector bundles and especially to Hermitian manifolds whose holomorphic tangent or cotangent bundles are globally generated to yield a family of new results on them (Theorems 4.1, 4.5, 4.6, and 4.11; Corollaries 4.2, 4.4, 4.7, and 4.12), some of which improve several related classical results.

Organization of this article. The rest of this article is organized as follows. In Section 2 we introduce necessary notation and symbols and then state our main results, Theorems 2.3 and 2.4. Then we give in Section 3 some examples where the Bott-Chern nonnegativity is satisfied. In Section 4 we apply our main results to obtain various related consequences, some of which improve several classical results, and the proof of Theorem 2.3 will be given in the last section, Section 5.

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2. MAIN RESULTS

Suppose that $(E^r, h) \longrightarrow M^n$ is a rank r Hermitian holomorphic vector bundle over a compact complex manifold M with $\dim_{\mathbb{C}} M = n$. This means that E^r is a rank r holomorphic vector bundle over M and is equipped with a Hermitian metric h on each fiber varying smoothly. We denote by ∇ the Chern connection of $(E^r, h) \longrightarrow M^n$, i.e., ∇ is the unique

connection on it compatible with both the complex structure and the Hermitian metric h . Then the curvature tensor R of ∇ is given by

$$R := \nabla^2 \in A^{1,1}(M; \text{Hom}(E, E)) = A^{1,1}(M; E^* \otimes E),$$

where $A^{1,1}(M; \cdot)$ is the set of complex-valued smooth $(1, 1)$ -forms with values in some bundle. If $\{e_1, \dots, e_r\}$ is a locally defined frame field of E , then

$$R(e_1, \dots, e_r) = (e_1, \dots, e_r)(\Omega_j^i),$$

where $\Omega := (\Omega_j^i)$ (i row, j column) is the curvature matrix with respect to $\{e_i\}$ whose entries are $(1, 1)$ -forms. If $\{\tilde{e}_i\}$ is another frame with

$$(\tilde{e}_1, \dots, \tilde{e}_r) = (e_1, \dots, e_r)P,$$

then the curvature matrix $\tilde{\Omega}$ with respect to $\{\tilde{e}_i\}$ is related to Ω by

$$(2.1) \quad \tilde{\Omega} = P^{-1}\Omega P.$$

It is well-known that the following $c_i(E, h)$ ($0 \leq i \leq n$):

$$(2.2) \quad \det\left(tI_r + \frac{\sqrt{-1}}{2\pi}(\Omega_j^i)\right) =: \sum_{i=0}^n c_i(E, h) \cdot t^{n-i}, \quad I_r : r \times r \text{ identity matrix},$$

are globally well-defined, real and closed (i, i) -forms and called the i -th Chern forms of (E, h) , which represent the Chern classes $c_i(E)$ of E .

The following definition was introduced by Bott and Chern ([BC65, p. 90]).

Definition 2.1 (Bott-Chern). A Hermitian holomorphic vector bundle $(E^r, h) \rightarrow M^n$ has nonnegative curvature, denoted by $h \geq_{\text{BC}} 0$, if for any point of M , there exist a *unitary* frame field around it and a matrix A with r rows whose entries are $(1, 0)$ -forms, such that the curvature matrix Ω under this unitary frame field satisfies

$$(2.3) \quad \Omega = A \wedge \overline{A}^t.$$

Here “ t ” denotes the transpose of a matrix.

Remark 2.2.

- (1) This definition is obviously inspired by the elementary fact that, for any matrix A whose entries are ordinary complex numbers, the matrix $A\overline{A}^t$ is Hermitian nonnegative definite.
- (2) This definition is independent of the unitary frame we choose as the transformation matrix P is a unitary matrix between them and thus from (2.1) we have

$$\tilde{\Omega} = P^{-1}\Omega P = P^{-1}(A \wedge \overline{A}^t)P = \overline{P}^t(A \wedge \overline{A}^t)P = (\overline{P}^t A) \wedge \overline{(P^t A)}^t.$$

- (3) Here we don't have any requirement on the number of columns of the matrix A , which may vary along the choice of the point on M .
- (4) We shall see in Example 3.1 that globally generated holomorphic vector bundles over compact complex manifolds form an important subclass of those admitting Bott-Chern nonnegative Hermitian metrics.

Before stating our first main result, we need some more notation. Following [FL83], we denote by $\Gamma(i, r)$ the set of all the partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_i)$ of weight i by nonnegative integers $\lambda_j \leq r$:

$$r \geq \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_i \geq 0, \quad \sum_{j=1}^i \lambda_j = i.$$

For each partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_i) \in \Gamma(i, r)$, a Schur polynomial

$$S_\lambda(c_1, \dots, c_r) \in \mathbb{Z}[c_1, \dots, c_r]$$

is attached to as follows:

$$S_\lambda(c_1, \dots, c_r) := \det(c_{\lambda_j - j + k})_{1 \leq j, k \leq i} \quad (j : \text{row}, k : \text{column})$$

$$= \begin{vmatrix} c_{\lambda_1} & c_{\lambda_1+1} & \cdots & c_{\lambda_1+i-1} \\ c_{\lambda_2-1} & c_{\lambda_2} & \cdots & c_{\lambda_2+i-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{\lambda_i-i+1} & c_{\lambda_i-i+2} & \cdots & c_{\lambda_i} \end{vmatrix}.$$

Note that we make the convention here that $c_0 = 1$ and $c_j = 0$ if $j < 0$ or $j > r$. In particular, we have

$$(2.4) \quad S_{(i, 0, \dots, 0)}(c_1, \dots, c_r) = c_i,$$

$$S_{(1, \dots, 1)}(c_1, \dots, c_r) = \text{dual Serge class of } c_1, \dots, c_r,$$

and

$$(2.5) \quad \begin{aligned} & S_{(i-j, j, 0, \dots, 0)}(c_1, \dots, c_r) \\ &= \begin{vmatrix} c_{i-j} & c_{i-j+1} & * & \cdots & * \\ c_{j-1} & c_j & * & \cdots & * \\ 0 & 0 & 1 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix} \quad (0 \leq j \leq \lfloor \frac{i}{2} \rfloor) \\ &= c_{i-j}c_j - c_{i-j+1}c_{j-1}. \end{aligned}$$

The expression (2.5) shall play a decisive role in establishing the lower and upper bounds of the Chern classes/numbers in our Theorem 2.4, which was partly inspired by [DPS94] as the special case

$$S_{(i-1, 1, 0, \dots, 0)}(c_1, \dots, c_r) = c_{i-1}c_1 - c_i$$

has been observed and applied in [DPS94, Coro. 2.6].

Recall that a real (p, p) -form φ ($1 \leq p \leq n$) on a compact complex manifold M^n is said to be *nonnegative*, denoted by $\varphi \geq 0$, if for any point $x \in M$ and any $(1, 0)$ -type tangent vectors X_1, \dots, X_p at x , we have

$$(-\sqrt{-1})^{p^2} \varphi(X_1, \dots, X_p, \overline{X_1}, \dots, \overline{X_p}) \geq 0.$$

In particular,

$$(2.6) \quad (\sqrt{-1})^{p^2} \sum \psi \wedge \overline{\psi} \geq 0, \quad \text{for any } (p, 0)\text{-form } \psi.$$

Two (p, p) -forms φ_1 and φ_2 are said to be $\varphi_1 \geq \varphi_2$ if $\varphi_1 - \varphi_2 \geq 0$. It is clear from the definition that the sum of two nonnegative forms of the same degree is nonnegative. But in general the product of two nonnegative forms needs not to be nonnegative (cf. [BP13]).

Nevertheless, the product of two nonnegative forms of the type (2.6) is still of the same type and thus nonnegative.

Bott and Chern noticed that if a Hermitian holomorphic vector bundle $(E^r, h) \rightarrow M^n$ with $h \geq_{BC} 0$, then all the Chern forms $c_i(E, h) \geq 0$ ($1 \leq i \leq n$) ([BC65, p. 91]). This fact was obtained again by Matsushima-Stoll in [MS73] in the case of globally generated holomorphic vector bundles by resorting to the universal holomorphic vector bundles of complex Grassmannians. The following theorem, which is our first main observation, states that indeed many more real forms involving in the Chern forms, including $c_i(E, h)$ themselves, are nonnegative.

Theorem 2.3. *Suppose that $(E^r, h) \rightarrow M^n$ is a Hermitian holomorphic vector bundles with $h \geq_{BC} 0$. Then the following closed real (i, i) -forms*

$$S_\lambda(c_1(E, h), \dots, c_r(E, h)), \quad (\forall \lambda \in \Gamma(i, r), \forall 1 \leq i \leq n)$$

(locally) have the type (2.6) and thus are nonnegative. In particular, by (2.4) and (2.5) the closed real (i, i) -forms

$$(2.7) \quad \begin{cases} c_i(E, h) & (1 \leq i \leq n) \\ c_{i-j}(E, h)c_j(E, h) - c_{i-j+1}(E, h)c_{j-1}(E, h) & (1 \leq i \leq n, 1 \leq j \leq [\frac{i}{2}]) \end{cases}$$

(locally) have the type (2.6) and hence are nonnegative.

For later simplicity we denote by

$$c_\lambda(E, h) := \prod_{j=1}^i c_{\lambda_j}(E, h), \quad \forall \lambda = (\lambda_1, \dots, \lambda_i) \in \Gamma(i, r),$$

$$c_\lambda[E] := \int_M \prod_{j=1}^n c_{\lambda_j}(E, h) \in \mathbb{Z}, \quad \forall \lambda = (\lambda_1, \dots, \lambda_n) \in \Gamma(n, r),$$

and

$$c_\lambda(E) := \prod_{j=1}^i c_{\lambda_j}(E) \in H^{2i}(M, \mathbb{Z}), \quad \forall \lambda = (\lambda_1, \dots, \lambda_i) \in \Gamma(i, r).$$

Now we are able to see that the nonnegativity of the forms in (2.7) in fact implies that the product of the Chern forms $c_i(E, h)$ and thus the Chern numbers of $E^r \rightarrow M^n$ can be bounded below and above as follows.

Theorem 2.4. *Suppose that $(E^r, h) \rightarrow M^n$ is a Hermitian holomorphic vector bundles with $h \geq_{BC} 0$. Then as (i, i) -forms $c_\lambda(E, h)$ are nonnegative and bounded below and above respectively by $c_i(E, h)$ and $c_1^i(E, h)$:*

$$0 \leq c_i(E, h) \leq c_\lambda(E, h) \leq [c_1(E, h)]^i, \quad \forall \lambda \in \Gamma(i, r), \forall 1 \leq i \leq n,$$

and consequently the Chern numbers satisfy

$$0 \leq c_n[E] \leq c_\lambda[E] \leq c_1^n[E], \quad \forall \lambda \in \Gamma(n, r).$$

Proof. For any $\lambda = (\lambda_1, \dots, \lambda_i) \in \Gamma(i, r)$, repeated use of (2.7) leads to

$$\begin{aligned} 0 \leq c_i(E, h) &\leq c_{i-1}(E, h)c_1(E, h) \\ &\leq c_{i-2}(E, h)c_2(E, h) \\ &\leq \dots \\ &\leq c_{i-\lambda_1}(E, h)c_{\lambda_1}(E, h), \quad \text{if } \lambda_1 \leq \lfloor \frac{i}{2} \rfloor, \end{aligned}$$

or

$$0 \leq c_i(E, h) \leq \dots \leq c_{\lambda_1}(E, h)c_{i-\lambda_1}(E, h), \quad \text{if } \lambda_1 > \lfloor \frac{i}{2} \rfloor.$$

This means that in any case we have

$$(2.8) \quad 0 \leq c_i(E, h) \leq c_{\lambda_1}(E, h)c_{i-\lambda_1}(E, h)$$

Similarly,

$$\begin{aligned} (2.9) \quad c_{i-\lambda_1}(E, h) &\leq c_{\lambda_2}(E, h)c_{i-\lambda_1-\lambda_2}(E, h), \\ c_{i-\lambda_1-\lambda_2}(E, h) &\leq c_{\lambda_3}(E, h)c_{i-\lambda_1-\lambda_2-\lambda_3}(E, h), \\ &\dots \end{aligned}$$

Note that all the nonnegative forms discussed here are of the type (2.6) by Theorem 2.3 and thus their products are still nonnegative:

$$(2.10) \quad 0 \leq c_i(E, h) \leq \prod_{j=1}^i c_{\lambda_j}(E, h) = c_\lambda(E, h).$$

On the other hand,

$$\begin{aligned} (2.11) \quad c_{\lambda_j}(E, h) &\leq c_{\lambda_j-1}(E, h)c_1(E, h) \\ &\leq c_{\lambda_j-2}(E, h)[c_1(E, h)]^2 \\ &\leq \dots \\ &\leq [c_1(E, h)]^{\lambda_j}. \end{aligned}$$

Therefore,

$$(2.12) \quad c_\lambda(E, h) = \prod_{j=1}^i c_{\lambda_j}(E, h) \leq \prod_{j=1}^i [c_1(E, h)]^{\lambda_j} = [c_1(E, h)]^i, \quad \forall \lambda \in \Gamma(i, r).$$

□

Remark 2.5. This theorem is in fact partly inspired by an observation in [DPS94, Coro. 2.6], where they noticed that $S_{(i-1,1,0,\dots,0)} = c_1c_{i-1} - c_i$ and thus gave the upper bound $c_1^i(E, h)$ for $c_\lambda(E, h)$ as we have done in (2.11) and (2.12) in the context of compact Kähler manifolds with nef holomorphic tangent bundles. Nevertheless, it is easy to see from (2.8), (2.9) and (2.10) that only the fact that “ $c_1c_{i-1} \geq c_i$ ” is *not* enough to derive the lower bound $c_i(E, h)$ as we need more. Here the beautiful part in our proof is that the expressions $S_{(i-j,j,0,\dots,0)}$ for general j are precisely enough to derive such a lower bound. Indeed we shall see in Section 4 that it is this lower bound which will play key roles in many related applications.

3. EXAMPLES

In this section we shall illustrate by some examples that in many important situations the Bott-Chern nonnegativity can be satisfied.

Recall that a holomorphic vector bundle $E \rightarrow M$ is called *globally generated* if the global holomorphic sections of E span the fiber over each point of M . If M is compact then $H^0(M, E)$, the complex vector space consisting of holomorphic sections of E , is finite-dimensional. Then the property of being globally generated implies that the following bundle sequence

$$(3.1) \quad 0 \rightarrow \ker(\varphi) \rightarrow M \times H^0(M, E) \xrightarrow{\varphi} E \rightarrow 0$$

$$(x, s) \mapsto (x, s(x)).$$

is exact. This means that a globally generated holomorphic vector bundle over a compact complex manifold can be realized as a quotient bundle of a trivial vector bundle. Then the conclusion that *any globally generated holomorphic vector bundle over a compact complex manifold admits a Bott-Chern nonnegative Hermitian metric* follows from the following general fact:

Example 3.1. If $(E^r, h) \rightarrow M$ is a Hermitian holomorphic vector bundle over a compact complex manifold M and S is an l -dimensional holomorphic subbundle of E ($l < r$) and thus $Q := E/S$ is a $(r - l)$ -dimensional holomorphic quotient bundle of E , i.e., we have the following short bundle exact sequence:

$$0 \rightarrow S \rightarrow E \rightarrow Q \rightarrow 0.$$

The Hermitian metric h on E naturally induces a metric on S and on Q . Bott and Chern noticed that ([BC65, p. 91]), under any unitary frame $\{e_1, \dots, e_r\}$ of E such that $\{e_1, \dots, e_l\}$ and $\{e_{l+1}, \dots, e_r\}$ are unitary frames of S and Q respectively, the curvature matrices of E and Q , denoted by Ω_E and Ω_Q , are related by (also cf. [GH78, p. 79], [Ko87, p. 23, (6.12)])

$$\Omega_E|_Q = \Omega_Q - A \wedge \overline{A}^t,$$

where $(\cdot)|_Q$ denotes the restriction to Q and A is a matrix whose entries are $(1, 0)$ -forms. This property is usually called “*curvature increases in holomorphic quotient bundles*”. This means that if E can be equipped with a *flat* Hermitian metric, which implies that $\Omega_E|_Q = (0)$ and is obviously satisfied by (3.1), then the quotient bundle Q admits a Bott-Chern nonnegative Hermitian metric.

Our principal concerns in this article are compact complex manifolds whose holomorphic tangent or cotangent bundles, denoted for simplicity in the sequel by T and T^* respectively, are globally generated. To each kind of them there is a well-known result, which we record in the following two examples for our later purpose.

Example 3.2. Recall that a compact complex manifold M is called *homogeneous* if the holomorphic automorphism group of M , denoted by $\text{Aut}(M)$, acts transitively on it. It is a well-known fact that *a compact connected complex manifold M has globally generated T_M if and only if it is homogeneous*. The reason is quite simple: T_M being globally generated is equivalent to the fact that each orbit of the action of $\text{Aut}(M)$ is open in M . Note that M is equal to a disjoint union of orbits. Then connectedness implies that there exists precisely one orbit, i.e., the action is homogenous. This particularly implies that all Hermitian symmetric spaces, complex flag manifolds and tori have globally generated holomorphic tangent bundles.

For compact complex manifolds whose cotangent bundles are globally generated we have the following well-known fact:

Example 3.3. *A compact connected Kähler manifold M has globally generated T_M^* if and only if M can be holomorphically immersed into some complex torus. This result should be due to Matsushima and Stoll ([MS73, p. 100] or [Ma92, p. 606]), at least to the author's best knowledge. The basic idea is to consider the Albanese map related to this Kähler manifold. A more compact proof can be found in [Sm76, p. 271]. Note that the term “globally generated” was called “ample” and “weakly ample” respectively in [MS73] and [Sm76]. Also note that a compact complex manifold holomorphically immersed into some torus is automatically Kähler as the restriction of the obvious Kähler metric on the complex torus is still Kähler. This implies that *any compact connected complex manifold M holomorphically immersed into some complex torus has globally generated T_M^* .**

The notion of *nefness* of holomorphic line/vector bundles over projective algebraic manifolds is well-known and has been extended to general compact complex manifolds by Demailly, which can be viewed as a limiting case of ampleness. For its definition and basic properties we refer to [De92, §1] or [DPS94, §1]. Before ending this section, we need to point out an implication, which will be used in the next section, that the Bott-Chern nonnegativity implies nefness. Indeed it turns out that Griffiths positivity implies ampleness ([Gr69, Theorem B], [Zh00, p. 206]). Taking the limiting case we know that Griffiths nonnegativity implies nefness. Then the above-mentioned implication follows from the fact that the Bott-Chern nonnegativity in fact implies the Griffiths nonnegativity. The latter fact must be well-known to experts, but we are not able to find a reference for this. So we record it in the following example and include a proof for the reader's convenience as well as for completeness.

Example 3.4. We have the following implications:

$$(3.2) \quad \text{Bott-Chern nonnegativity} \implies \text{Griffiths nonnegativity} \implies \text{nefness}.$$

Proof. We show the first implication in (3.2). Under a local coordinate (z^1, \dots, z^n) , the entries Ω_j^i in the curvature matrix Ω with respect to some unitary frame field can be written as

$$\Omega_j^i = R_{jpq}^i dz^p \wedge d\bar{z}^q.$$

The Hermitian metric h on E is called *Griffiths nonnegative* ([Gr69, p. 181]) if, at any point of M , we have

$$\sum_{i,j,p,q} R_{jpq}^i \xi^j \eta^p \bar{\xi}^i \bar{\eta}^q \geq 0$$

for any $\xi = (\xi^1, \dots, \xi^r) \in \mathbb{C}^r$, and any $\eta = (\eta^1, \dots, \eta^n) \in \mathbb{C}^n$. Now we assume that $h \geq_{\text{BC}} 0$, i.e.,

$$(\Omega_j^i) = A \wedge \bar{A}^t, \quad A = \left(\sum_p T_{ij}^{(p)} dz^p \right).$$

This implies that

$$R_{jpq}^i = \sum_k T_{ik}^{(p)} \overline{T_{jk}^{(q)}}$$

and thus

$$\begin{aligned} \sum_{i,j,p,q} R_{j pq}^i \xi^j \eta^p \overline{\xi^i \eta^q} &= \sum_{k,i,j,p,q} T_{ik}^{(p)} \overline{T_{jk}^{(q)}} \xi^j \eta^p \overline{\xi^i \eta^q} = \sum_{k,i,j,p,q} (T_{ik}^{(p)} \overline{\xi^i \eta^p}) (\overline{T_{jk}^{(q)}} \xi^j \eta^q) \\ &= \sum_k \left| \sum_{i,p} T_{ik}^{(p)} \overline{\xi^i \eta^p} \right|^2 \geq 0. \end{aligned}$$

□

4. APPLICATIONS

In this section we give three applications related to Theorem 2.4. The first two applications are more or less quite direct, while the third application are subtle and more things are involved in its proof.

As Bott-Chern has observed that ([BC65, p. 91]) the Chern forms of globally generated holomorphic vector bundles are all nonnegative and thus it is immediate that their Chern numbers are nonnegative.

Now the *first direct application* of our Theorem 2.4 to Bott-Chern nonnegative Hermitian holomorphic vector bundles yields the following improvements of the above-mentioned nonnegativity of Chern numbers by giving lower and upper bounds.

Theorem 4.1. *Suppose $(E^r, h) \rightarrow M^n$ is a Hermitian holomorphic vector bundle over a compact complex manifold with $h \geq_{BC} 0$. Then all their Chern numbers are nonnegative and bounded below and above by $c_n[E]$ and $c_1^n[E]$:*

$$(4.1) \quad 0 \leq c_n[E] \leq c_\lambda[E] \leq c_1^n[E], \quad \forall \lambda \in \Gamma(n, r).$$

This particularly holds for globally generated holomorphic vector bundles over compact complex manifolds.

In particular, if the Chern number $c_1^n[E] = 0$, then all the Chern numbers of $E \rightarrow M$ vanish.

Compact connected homogeneous complex manifolds have been well-studied and it is also a classical result that their Chern numbers are nonnegative ([Go54], [BR62], [GR62], [MS73, Theorem 6.2]). The nonnegativity of the signed Chern numbers of compact connected complex manifolds holomorphically immersed into complex tori has also been obtained by Matsushima-Stoll ([MS73, Theorem 6.5], [Ma92, p. 607]). These examples, as we have seen in Examples 3.2 and 3.3, satisfy the assumptions in Theorem 4.1 and thus we have the following corollary, which also improves the nonnegativity results for Chern numbers/signed Chern numbers by giving lower and upper bounds respectively.

Corollary 4.2.

- (1) The Chern numbers of any compact connected homogeneous complex manifold M^n are nonnegative and bounded below and above by $c_n[M]$ and $c_1^n[M]$:

$$(4.2) \quad 0 \leq c_n[M] \leq c_\lambda[M] \leq c_1^n[M], \quad \forall \lambda \in \Gamma(n, n).$$

- (2) The signed Chern numbers of any compact connected complex manifold M^n whose holomorphic cotangent bundle is globally generated are nonnegative and bounded

below and above by $(-1)^n c_n[M]$ and $(-1)^n c_1^n[M]$:

$$(4.3) \quad 0 \leq (-1)^n c_n[M] \leq (-1)^n c_\lambda[M] \leq (-1)^n c_1^n[M], \quad \forall \lambda \in \Gamma(n, n).$$

This particularly holds for compact connected complex manifolds holomorphically immersed into complex tori.

In particular, in these two cases, if the Chern number $c_1^n[M] = 0$, then all the Chern numbers of M vanish.

Remark 4.3. The consideration of compact connected Kähler manifolds with globally generated holomorphic cotangent bundles can be traced back to Bochner ([Bo50]), where he showed the nonnegativity of the signed Euler characteristic.

As mentioned in the Introduction, Demailly, Peternell and Schneider has showed the non-negativity of the Schur polynomials in the Chern classes for nef holomorphic vector bundles over compact Kähler manifolds ([DPS94, Theorem 2.5]) and obtained the upper bound c_1^i for any c_λ , $\lambda \in \Gamma(i, r)$. Now by our arguments and remarks in Theorem 2.4 their result can be complemented by adding a lower bound as follows.

Corollary 4.4. Suppose $E^r \rightarrow (M, \omega)$ is a nef holomorphic vector bundle over a compact Kähler manifold with Kähler form ω . Then we have

$$0 \leq \int_M c_i(E) \wedge [\omega]^{n-i} \leq \int_M c_\lambda(E) \wedge [\omega]^{n-i} \leq \int_M c_1^i(E) \wedge [\omega]^{n-i}, \quad \forall \lambda \in \Gamma(i, r).$$

In particular, if the Chern number $c_1^n[E] = 0$ (resp. $c_n[E] > 0$), then all the Chern numbers of E vanish (resp. are positive).

Our second application is concerned with the projectivity of compact complex manifolds equipped with globally generated holomorphic vector bundles and compact connected homogeneous complex manifolds, and its relationship with the non-vanishing of Chern classes/numbers. First let us recall some more notation. For a compact complex manifold M^n , the *algebraic dimension* of M , denoted by $a(M)$, is defined to be the maximal number of global meromorphic functions on M that can be algebraically independent. In other words, $a(M)$ is the transcendental degree of function field of M over \mathbb{C} . $a(M)$ satisfies $0 \leq a(M) \leq n$. A compact complex manifold M^n is called *Moishezon* if the algebraic dimension $a(M) = n$, in which case it was investigated in detail by Moishezon ([Mo66]) and a deep theorem of his states that ([Mo66])

$$(4.4) \quad \text{a Moishezon manifold is Kähler if and only if it is projective algebraic.}$$

The starting points of our second application are two basic results due to Matsushima-Stoll and Demailly respectively. A major result of Matsushima-Stoll says that ([MS73, Theorems 5.5, 5.6] or [Ma92, p. 600]) if a globally generated holomorphic vector bundle $E^r \rightarrow M^n$ satisfies $0 \neq c_\lambda(E) \in H^{2i}(M, \mathbb{Z})$ for some $\lambda \in \Gamma(i, r)$, then the algebraic dimension $a(M) \geq i$. In particular,

$$(4.5) \quad \text{if some Chern number of a globally generated holomorphic vector bundle over } M \text{ is nonzero, then } M \text{ is Moishezon.}$$

This result, together with Moishezon's result (4.4), leads to the main theorem of [MS73] ([MS73, p. 94, Main Theorem] or [Ma92, p. 600]) saying that if some Chern number of a globally generated holomorphic vector bundle over a compact Kähler manifold is nonzero,

then this manifold is projective algebraic. Another basic result of Demailly ([De92, Corollary 1.6] or [DPS94, Theorem 4.1]) tells us that

(4.6) *a Moishezon compact complex manifold with nef holomorphic tangent bundle is projective algebraic.*

A combination of the results (4.4), (4.5), and (4.1) in our Theorem 4.1 in turn leads to the following improvement of (4.5).

Theorem 4.5. *Suppose $E^r \rightarrow M^n$ is a globally generated holomorphic vector bundle over a compact complex manifold (resp. compact Kähler manifold). Then we have the following implications:*

$$\begin{aligned} & \text{The Chern number } c_1^n[E] \neq 0 \\ \iff & \text{Some Chern numbers of } E \rightarrow M \text{ is nonzero} \\ \implies & M \text{ is a Moishezon (resp. projective algebraic) manifold.} \end{aligned}$$

Compact connected homogeneous complex manifolds have been well-studied and it has been known that (cf. [MS73, Theorem 6.2] or [Ma92, p. 601]) they are projective algebraic provided that their Euler characteristic are nonzero (hence positive by (4.2) in Corollary 4.2). Our following result is a refinement of this classical fact, and, moreover its proof is built on the aforementioned two basic results and thus different from the original one.

Theorem 4.6. *Suppose M^n is a compact connected homogeneous complex manifold. Then we have the following implications:*

$$\begin{aligned} & \text{The Chern number } c_1^n[M] \neq 0. \\ (4.7) \quad \iff & \text{Some Chern numbers of } M \text{ is nonzero.} \\ \implies & M \text{ is a projective algebraic manifold.} \end{aligned}$$

Proof. If some Chern number is nonzero and thus positive, which is equivalent to the positivity of the Chern number $c_1^n[M]$ by virtue of (4.2), then M is Moishezon due to (4.5) as the holomorphic tangent bundle of M is globally generated. Now the property of being globally generated implies the existence of a Bott-Chern nonnegative Hermitian metric on its holomorphic tangent bundle by Example 3.1 and thus implies nefness by (3.2) in Example 3.4. Then the fact of M being projective algebraic follows from Demailly's result (4.6). \square

The converse part to (4.7) is *not* true in general, i.e., even if a compact connected homogeneous complex manifold is projective algebraic, it may happen that its all Chern numbers vanish. For instance, the product of an abelian variety over \mathbb{C} and a projective-rational manifold is projective algebraic but its Chern numbers are all zero due to the existence of an abelian variety. Indeed, a classical result of Borel and Remmert tells us that ([BR62])

(4.8) *any compact connected homogeneous Kähler manifold is the product of a complex torus and a projective-rational manifold.*

Theorem 4.6 and (4.8) implies that the converse part to (4.7) is still true if we further impose *simple-connectedness* on the manifolds in question:

Corollary 4.7. *Suppose M^n is a simply-connected compact connected homogeneous complex manifold. Then the following four conditions are equivalent:*

- (1) The Chern number $c_1^n[M] \neq 0$.
- (2) Some Chern numbers of M is nonzero.
- (3) M is a projective algebraic manifold.
- (4) M is a Kähler manifold.

Remark 4.8.

- (1) The manifolds considered in Corollary 4.7 were called C -spaces by Wang ([Wa54]) and investigated in detail by many authors ([BR62], [Go54], [GR62], [Wa54] etc.). However, the contents in our Corollary 4.7 should be completely new, at least to the author's best knowledge.
- (2) In the conclusions in Theorems 4.5, 4.6 and Corollary 4.7, when the Chern numbers in consideration are nonzero, they must be positive by (4.2) in Corollary 4.2.
- (3) There are homogeneous complex structures on the product of odd-dimensional spheres: $M_{p,q} := S^{2p+1} \times S^{2q+1}$ ($(p, q) \neq (0, 0)$), which are called Calabi-Eckmann manifolds ([CE53]) and whose Chern numbers are all zero as they represent the zero elements in the unitary cobordism ring. $M_{p,q}$ are indeed non-Kähler as the second Betti number is zero and so these examples match our Corollary 4.7.

Our third application is to give an in-depth investigation on the structure of compact connected complex manifolds holomorphically immersed into complex tori in detail. As we have seen in Example 3.3, these manifolds are precisely those compact connected Kähler manifolds whose holomorphic cotangent bundles are globally generated. The structure of this kind of manifolds was first investigated by Matsushima and his coauthors ([HM75], [Ma74], [MS73]), Yau ([Ya74, Chapter 3]) and Smyth ([Sm76]) etc around the same time.

Before continuing, let us digress to recall the notion of Kodaira dimension $\kappa(M)$ for compact complex manifolds M ([Zh00, p. 132]), which has several equivalent definitions. Here for our later purpose we adopt the following one. Denote by K_M the canonical line bundle of M . It turns out that $\dim_{\mathbb{C}} H^0(M, K_M^{\otimes m})$, the complex dimension of the holomorphic sections of $K_M^{\otimes m}$, has the following property: either $H^0(M, K_M^{\otimes m}) = 0$ for all $m \leq 1$ or there exists an integer $0 \leq \kappa(M) \leq \dim_{\mathbb{C}}(M)$ and constants $0 \leq C_1 < C_2$ such that

$$(4.9) \quad C_1 m^{\kappa(M)} \leq \dim_{\mathbb{C}} H^0(M, K_M^{\otimes m}) \leq C_2 m^{\kappa(M)}, \quad m \gg 0.$$

In the former case $\kappa(M) := -\infty$. M is called of *general type* if $\kappa(M) = \dim_{\mathbb{C}}(M)$. It is an elementary fact between the Kodaira dimension $\kappa(M)$ and the algebraic dimension $a(M)$ that ([Zh00, p. 135]):

$$(4.10) \quad \kappa(M) \leq a(M) \leq \dim_{\mathbb{C}}(M).$$

Denote by $\text{Aut}_0(M)$ the identity component of the holomorphic automorphism group of a compact complex manifold M , which is a connected complex Lie group. The most fundamental structure of compact connected complex manifolds holomorphically immersed into complex tori is the following result.

Theorem 4.9. ([Ma74, Prop. 1], [Ya74, Th. 5], [Sm76, Th. 1]) *Suppose M^n is a compact connected complex manifold holomorphically immersed into some complex torus with $\text{Aut}_0(M) \neq \{0\}$. Then $\text{Aut}_0(M)$ is a complex torus and acts freely on M . Moreover, the quotient manifold $N := M/\text{Aut}_0(M)$ is also a compact connected complex manifold and can be holomorphically immersed into some complex torus and $\text{Aut}_0(N) = \{0\}$. Consequently, M is*

a holomorphic principal complex torus bundle over some compact connected complex manifold N also holomorphically immersed into some complex torus with $\text{Aut}_0(N) = \{0\}$.

This structure theorem has several applications in [Ma74], [Ya74] and [Sm76]. Let us record two of them related to our next application in the following corollary for our later citation.

Corollary 4.10. Suppose M^n is a compact connected complex manifold holomorphically immersed into some complex torus. Then

- (1) ([Ya74, p. 238, Coro.]) M is a (possibly trivial) torus bundle over a compact Kähler manifold of general type.
 - (2) ([Sm76, p. 278, Coro. 1]) the following three conditions are equivalent:
- (4.11) $\text{Aut}_0(M) = \{0\} \iff c_n[M] \neq 0 \iff c_1^n[M] \neq 0.$

We can now give our third application, which relates $\text{Aut}_0(M) = \{0\}$, all the Chern numbers, $\kappa(M)$ and torus bundle structure in a more explicit manner and particularly improves Corollary 4.10.

Theorem 4.11. Suppose M^n is a compact connected complex manifold holomorphically immersed into some complex torus, or equivalently, M^n is a compact connected Kähler manifold with globally generated holomorphic cotangent bundle. Then the following four conditions are equivalent:

- (1) $\text{Aut}_0(M) = \{0\}$,
- (2) all the signed Chern numbers of M are strictly positive,
- (3) the Kodaira dimension $\kappa(M) = n$, i.e., M is of general type,
- (4) M cannot be realized as a total space of some nontrivial torus bundle,

and the following four conditions are equivalent:

- (5) $\text{Aut}_0(M) \neq \{0\}$,
- (6) all the Chern numbers of M vanish,
- (7) the Kodaira dimension $\kappa(M) < n$,
- (8) M is a nontrivial holomorphic principal complex torus bundle over a compact connected Kähler manifold of general type.

Moreover, under (any one of) the first four equivalent conditions, M is a projective algebraic manifold.

A direct corollary of Theorem 4.11 is the following result, which tells us that whether or not the Chern numbers vanish is simultaneous.

Corollary 4.12. Suppose M^n is a compact connected complex manifold holomorphically immersed into some complex torus, or equivalently, M^n is a compact connected Kähler manifold with globally generated holomorphic cotangent bundle. If some Chern number of M is nonzero (resp. vanishes), then all the Chern numbers of M are nonzero and have the sign $(-1)^n$ (resp. vanish), in which case $\text{Aut}_0(M)$ is trivial (resp. non-trivial) and M is of general type (resp. the Kodaira dimension $\kappa(M) < n$).

Proof of Theorem 4.11.

First we show that the condition (2) implies that M is projective algebraic and thus prove the last conclusion in Theorem 4.11. Indeed, (4.5) says that M is Moishezon. Then M being projective algebraic follows from (4.4) and the fact that M is Kähler.

“(1) \Leftrightarrow (2)” follows from (4.11) and (4.3) in our Corollary 4.2.

“(2) \Leftrightarrow (4)” : If M can be realized as a total space of some nontrivial torus bundle, then $c_n[M] = 0$ as the Euler characteristic is multiplicative for fiber bundles, which implies that “(2) \Rightarrow (4)”. “(4) \Rightarrow (1)” follows from Theorem 4.9.

“(2) \Rightarrow (3)” : The proof is a combination of a Kodaira-type vanishing theorem and the Hirzebruch-Riemann-Roch theorem. By the definition of M we can choose a Hermitian metric h on the holomorphic cotangent bundle T_M^* such that $(h, T_M^*) \geq_{\text{BC}} 0$. Then (2.7) in Theorem 2.3 tells us that the first Chern form $c_1(T_M^*, h) \geq 0$. This means that the Chern class

$$(4.12) \quad c_1(K_M^{\otimes m}) = mc_1(K_M) = m[c_1(T_M^*, h)] \geq 0, \quad m \geq 1.$$

Condition (2) tells us that

$$(4.13) \quad \int_M (c_1(K_M^{\otimes m}))^n = m^n (-1)^n c_n[M] > 0, \quad m \geq 1.$$

We have shown above that M is projective algebraic under the condition (2). Then a combination of (4.12), (4.13) and M being projective algebraic, due to the Kodaira-Kawamata-Viehweg vanishing theorem for projective algebraic manifolds ([Ko87, p. 74, (3.10)]), yields

$$(4.14) \quad H^q(M, K_M^{\otimes m}) = 0, \quad q \geq 1, \quad m \geq 2.$$

Then the Hirzebruch-Riemann-Roch theorem tells us that ([Hi66])

$$\begin{aligned} \dim_{\mathbb{C}} H^0(M, K_M^{\otimes m}) &= \sum_{q=0}^n (-1)^q \dim_{\mathbb{C}} H^q(M, K_M^{\otimes m}) \quad ((4.14)) \\ &= \int_M [\text{Td}(M) \cdot \text{ch}(K_M^{\otimes m})] \quad (\text{Td: Todd class, ch: Chern character}) \\ &= \int_M \left\{ \left[1 + \frac{1}{2} c_1(M) + \cdots \right] \cdot \exp \left[-mc_1(M) \right] \right\} \\ &= \frac{(-1)^n c_1^n[M]}{n!} m^n + O(m^{n-1}), \quad (m \gg 0) \end{aligned}$$

which, together with $(-1)^n c_1^n[M] > 0$ under the condition (2), implies (4.9) with the Kodaira dimension $\kappa(M) = n$, i.e., M is of general type.

“(3) \Rightarrow (2)” : We first assert that M is projective algebraic under the condition (3). Indeed, $\kappa(M) = n$ implies that the algebraic dimension $a(M) = n$ by (4.10) and thus M is Moishezon. This implies that M is projective algebraic by (4.4) as M is Kähler. If on the contrary the Euler characteristic $c_n[M] = 0$, then a result of Howard and Matsushima ([HM75, Th. 6] or [Ma92, p. 655]) says that M admits a holomorphic one-form with no zero points. However, a recent remarkable result of Popa and Schnell ([PS14]) tells us that every holomorphic one-form on a projective algebraic manifold of general type has at least one zero point, which leads to a contradiction. This means that $c_n[M] \neq 0$ and thus proves condition (2) by (4.3).

“(5) \Leftrightarrow (6)” also follows from (4.3) in Corollary 4.2 and (4.11).

“(5) \Leftrightarrow (7)” follows directly from “(1) \Leftrightarrow (3)” as they are equivalent.

“(5) \Rightarrow (8)”: Theorem 4.9 tells us, under the condition $\text{Aut}_0(M) \neq \{0\}$, that M is a nontrivial holomorphic principal complex torus bundle over a compact connected complex manifold N , where N can also be holomorphically immersed some complex torus with $\text{Aut}_0(N) = \{0\}$. Then N is Kähler and “(1) \Leftrightarrow (3)” implies that N is of general type.

“(8) \Rightarrow (5)”: Condition (8) implies that the Euler characteristic number $c_n[M] = 0$ and thus $\text{Aut}_0(M) \neq \{0\}$ by “(1) \Leftrightarrow (2)”.

This completes the proof of Theorem 4.11.

Remark 4.13. We can easily see from Theorem 4.9 and the above proof that the condition (8) in Theorem 4.11 can be strengthened to

$$(4.15) \quad \begin{aligned} & \text{“}M \text{ is a nontrivial holomorphic principal complex torus bundle over some compact} \\ & \text{connected Kähler manifold } N \text{ with globally generated } T_N^* \text{ and trivial } \text{Aut}_0(N)\text{”} \end{aligned}$$

or weakened to

$$(4.16) \quad \text{“}M \text{ can be realized as a total space of some nontrivial torus bundle”,}$$

or can be stated as some condition, as we have done in Theorem 4.11, provided that

$$\text{“(4.15)} \Rightarrow \text{this condition} \Rightarrow \text{(4.16)}\text{”}.$$

The reason for our current choice of condition (8) in Theorem 4.11 is to match Yau’s result in Corollary 4.10. The condition (4) in Theorem 4.11 can be similarly treated.

5. PROOF OF THEOREM 2.3

In this last section we shall prove Theorem 2.3 and complete this article.

Let us begin by recalling some well-known facts about $GL_r(\mathbb{C})$ -invariant polynomial functions. In what follows $GL_r(\mathbb{C})$ and $M_r(\mathbb{C})$ denote respectively the general linear group of order r and the $r \times r$ matrix group, both over \mathbb{C} . A map $f : M_r(\mathbb{C}) \rightarrow \mathbb{C}$ is called a $GL_r(\mathbb{C})$ -invariant polynomial function of homogeneous degree i if f can be written as

$$(5.1) \quad f(A = (T_{\alpha\beta})) = \sum_{1 \leq \alpha_j, \beta_j \leq r} p_{\alpha_1 \dots \alpha_i, \beta_1 \dots \beta_i} T_{\alpha_1 \beta_1} \cdots T_{\alpha_i \beta_i}$$

for $p_{\alpha_1 \dots \alpha_i, \beta_1 \dots \beta_i} \in \mathbb{C}$ and satisfies

$$(5.2) \quad f(BAB^{-1}) = f(A), \quad \forall A \in M_r(\mathbb{C}), \forall B \in GL_r(\mathbb{C}).$$

Denote by I_i the complex linear space consisting of all $GL_r(\mathbb{C})$ -invariant polynomial function of homogeneous degree i :

$$I_i := \{f : M_r(\mathbb{C}) \rightarrow \mathbb{C} \mid f \text{ satisfy (5.1) and (5.2)}\}$$

and the graded ring

$$I := \bigoplus_{i \geq 0} I_i.$$

If we set

$$\det(tI_r + A) =: \sum_{i=0}^r c_i(A) \cdot t^{r-i},$$

then these $c_i(\cdot)$ are $GL_r(\mathbb{C})$ -invariant polynomial functions of homogeneous degree i . In particular, $c_0(A) = 1$, $c_1(A) = \text{trace}(A)$ and $c_n(A) = \det(A)$. It is well-known that the graded ring I is multiplicatively generated by c_1, \dots, c_r , i.e.,

$$I = \mathbb{C}[c_1, \dots, c_r].$$

Now if $(E^r, h) \longrightarrow M$ is a Hermitian holomorphic vector bundle and (Ω_j^i) is the curvature matrix of its Chern connection, then (recall (2.1) and (2.2))

$$c_i\left(\frac{\sqrt{-1}}{2\pi}(\Omega_j^i)\right) =: c_i(E, h), \quad (0 \leq i \leq r)$$

are globally defined, closed, real-valued (i, i) -forms over M and represent the i -th Chern classes of $E \longrightarrow M$. Thus we have a natural graded ring homomorphism φ sending c_i to $c_i(E, h)$:

$$(5.3) \quad I = \bigoplus_{i \geq 0} I_i = \mathbb{C}[c_1, \dots, c_r] \xrightarrow{\varphi} \mathbb{C}[c_1(E, h), \dots, c_r(E, h)]$$

$$f \longmapsto f\left(\frac{\sqrt{-1}}{2\pi}(\Omega_j^i)\right).$$

Griffiths first showed that ([Gr69, p. 242, (5.6)]) any $f \in I_i$ can be written, which may not be unique, in the form

$$(5.4) \quad f(A = (T_{ij})) = \sum_{\pi, \tau \in S_i, \rho = (\rho_1, \dots, \rho_i) \in [1, r]^i} p_{\rho, \pi, \tau} T_{\rho_{\pi(1)} \rho_{\tau(1)}} \cdots T_{\rho_{\pi(i)} \rho_{\tau(i)}}, \quad p_{\rho, \pi, \tau} \in \mathbb{C},$$

where S_i denotes the permutation group on i objects.

The following definition was introduced by Griffiths ([Gr69, p. 242, (5.9)]).

Definition 5.1 (Griffiths). A polynomial function $f \in I_i$ is called *Griffiths positive* if it can be expressed in the form (5.4) with

$$(5.5) \quad p_{\rho, \pi, \tau} = \sum_{j \in I} \lambda_{\rho, j} q_{\rho, j, \pi} \bar{q}_{\rho, j, \tau}, \quad \forall \rho, \pi, \tau,$$

for some real numbers $\lambda_{\rho, j} \geq 0$, some complex numbers $q_{\rho, j, \pi}$ and some finite set J . Denote by

$$\Pi_i := \{f \in I_i \mid f \text{ are Griffiths positive}\}.$$

The following key fact relating Griffiths positive polynomials and Schur polynomials was observed in [FL83, p. 54, Prop. A.3], whose proof is built purely on the representation theory ([FL83, p. 56]).

Proposition 5.2 (Fulton-Lazarsfeld).

$$(5.6) \quad \left\{ \sum_{\lambda \in \Gamma(i, r)} a_\lambda S_\lambda(c_1, \dots, c_r) \mid \text{all } a_\lambda \geq 0 \right\} \subset \Pi_i.$$

Now we are in the position to prove Theorem 2.3.

Proof. Assume that the Hermitian holomorphic vector bundle $(E^r, h) \rightarrow M^n$ satisfies $h \geq_{\text{BC}} 0$, i.e., under the local coordinates (z^1, \dots, z^n) , the curvature matrix (Ω_j^i) with respect to some unitary frame field can be written in the following form

$$(\Omega_j^i) = A \wedge \overline{A^t}, \quad A = \left(\sum_p T_{ij}^{(p)} dz^p \right).$$

This implies that

$$(5.7) \quad \Omega_j^i = \sum_{k,p,q} T_{ik}^{(p)} \overline{T_{jk}^{(q)}} dz^p \wedge d\overline{z^q}.$$

Therefore, for each $\lambda \in \Gamma(i, r)$ and $1 \leq i \leq n$, we have

$$\begin{aligned} & S_\lambda(c_1(E, h), \dots, c_r(E, h)) \\ &= \varphi(S_\lambda(c_1, \dots, c_r)) \quad ((5.3)) \\ &= S_\lambda\left(c_1\left(\frac{\sqrt{-1}}{2\pi}(\Omega_j^i)\right), \dots, c_r\left(\frac{\sqrt{-1}}{2\pi}(\Omega_j^i)\right)\right) \quad ((5.3)) \\ &= \left(\frac{\sqrt{-1}}{2\pi}\right)^i \sum \lambda_{\rho,j} q_{\rho,j,\pi} \overline{q}_{\rho,j,\tau} \Omega_{\rho_{\tau(1)}k_1}^{\rho_{\pi(1)}} \cdots \Omega_{\rho_{\tau(i)}k_i}^{\rho_{\pi(i)}} \quad ((5.4), (5.5), (5.6)) \\ &= \left(\frac{\sqrt{-1}}{2\pi}\right)^i \sum \lambda_{\rho,j} q_{\rho,j,\pi} \overline{q}_{\rho,j,\tau} T_{\rho_{\tau(1)}k_1}^{(r_1)} \overline{T_{\rho_{\tau(1)}k_1}^{(s_1)}} dz^{r_1} \wedge d\overline{z^{s_1}} \cdots T_{\rho_{\tau(i)}k_i}^{(r_i)} \overline{T_{\rho_{\tau(i)}k_i}^{(s_i)}} dz^{r_i} \wedge d\overline{z^{s_i}} \quad ((5.7)) \\ &= \left(\frac{\sqrt{-1}}{2\pi}\right)^i (-1)^{\frac{i(i-1)}{2}} \sum_{\rho,j} \lambda_{\rho,j} \left\{ \left[\sum_{\pi,k,r} q_{\rho,j,\pi} T_{\rho_{\pi(1)}k_1}^{(r_1)} \cdots T_{\rho_{\pi(i)}k_i}^{(r_i)} dz^{r_1} \wedge \cdots \wedge dz^{r_i} \right] \right. \\ &\quad \left. \wedge \left[\sum_{\tau,k,s} \overline{q_{\rho,j,\tau} T_{\rho_{\tau(1)}k_1}^{(s_1)} \cdots T_{\rho_{\tau(i)}k_i}^{(s_i)}} dz^{s_1} \wedge \cdots \wedge dz^{s_i} \right] \right\} \\ &=: \frac{(\sqrt{-1})^{i^2}}{(2\pi)^i} \sum_{\rho,j} \lambda_{\rho,j} \psi_{\rho,j} \wedge \overline{\psi_{\rho,j}}. \end{aligned}$$

Now $\psi_{\rho,j}$ are $(i, 0)$ -forms and by the definition $\lambda_{\rho,j} \geq 0$ (recall (5.5)), which means that the last expression is of the type (2.6) and hence a nonnegative (i, i) -form. This gives the desired proof and thus completes this article. \square

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